The Banach Contraction Principle: Generalizations and Applications

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Outline

Fixed Point Theory

A fixed point of a mapping T from a set into itself is an element x such that $T(x) = x$.

- The Banach Fixed Point Theorem was not the first theorem connected with fixed points.
- One of the first theorems were formulated by Henri Poincaré in 1886.
- In 1909 Luitzen E. J. Brouwer proved that any continuous function f from a closed ball in $\mathbb{R}^n$ into itself has at least one fixed point.
- This proof was presented for $n = 3$.

- A year later Hadamard showed the generalization of this theorem for arbitrary n.
- In 1912 Brouwer gave another proof for this generalization.

The Brouwer theorem was a non-constructive result. It was only existential. It said about the existence of such point but did not explain how to obtain it. The proof of the Banach Fixed Point Theorem was constructive. It gives the existence and uniqueness of a fixed point and convergence of the sequence of successive approximations to a solution of the problem.

- Banach's doctoral thesis defended on June 24, 1920 was the very beginning of functional analysis,
- Banach's book "Théorie des opérations linéaires (1932)" was the beginning of linear functional analysis.

S. Banach, Théorie des Opérations Linéaires. Monogr. Mat., t. **I**, Warszawa, 1932.

Fixed Point Theory is divided into three major areas:

- Topological fixed point theory
- Metric fixed point theory
- Discrete fixed point theory

Historical Fixed Point Theorems

Historically the boundary lines between the three areas were defined by the discovery of three major theorems:

- Brouwer (1912)
- Banach (1922)
- Tarski (1955)

1. L.E.J. Brouwer, von Mannigfaltigkeiten. Math. Ann. **71** (1912), 97-115
2. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundam. Math. **3** (1922), 133-181
3. A. Tarski, A lattice theoretical fixed point theorem and its applications. Pacific J. Math. 5 (1955), 285-309

Theorem (Brouwer, version 1)

Theorem Every continuous mapping from a closed ball in $\mathbb{R}^n$ into itself has a fixed point.

Example: Any continuous map $T : [a, b] \to [a, b]$ has at least one fixed point.

Let the map $F(x) = T(x) - x, \; x \in [a, b]$. Then

$$F(a)F(b) \leq 0.$$

Hence there exists at least one element $x^* \in [a, b]$ such that $F(x^*) = 0$, which is a fixed point of T.

Theorem (Brouwer, version 2)

Let $K \subset \mathbb{R}^n$ be compact, convex, and nonempty. Any continuous operator $T : K \to K$ has at least one fixed point.

The following counter examples show the essentials of each assumption in the Brouwer fixed point theorem.

1. Let $f : (0,1) \to (0,1)$ be defined by $f(x) = \frac{x}{2}$. This map is continuous and maps $(0,1)$ into itself, but the domain is not compact
2. $K = \mathbb{R}$ and $T : K \to K,\ T(x) = x + 1$. T is continuous, and K is convex, nonempty, but not compact. No fixed point.
3. Let $D = \{(x,0) | x \in [-1,-0.5] \cup [0.5,1]\}$. Define the function $f(x,0) = (-x,0)$. The domain is compact, but not convex (disconnected).

Let the function $f : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1), \\ 0, & \text{if } x = 1. \end{cases}$$

This function is not continuous at $x = 1$,

Discrete fixed point theory

Discrete fixed point theory came from Alfred Tarski in 1955. Tarski fixed point theorem is a discrete fixed point result. Note that a discrete fixed point is a fixed point for functions defined on finite sets. Recall that a lattice is a partially ordered set $(M, \leq)$ in which every pair of elements $a, b \in M$ has a least upper bound (called the join) noted $a \vee b = \max\{a, b\}$ and a greatest lower bound (called the meet), denoted $a \wedge b = \min\{a, b\}$

An Example

Let X be a set and $P(X)$ the family of all subsets of X ordered by the inclusion. Let $A, B \in P(X)$, then

$$A \wedge B = A \cap B, \text{ and } A \vee B = A \cup B.$$

This is a Boolean lattice.
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An Example

Let $\mathbb{N}$ be the set of positive integers ordered by the divisibility. Let $a, b \in \mathbb{N}$, then

$$a \wedge b = GCD(a, b), \text{ and } a \vee b = LCM(a, b).$$

This is an unbounded lattice.

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Theorem (Tarski, 1955)

If T is a monotone function on a nonempty complete lattice, then

1. The set of fixed points of T is nonempty,
2. the set of fixed points of T forms a nonempty complete lattice.
3. T has a least fixed point and a greatest fixed point.

A. Tarski, A lattice theoretical fixed point theorem and its applications. Pacific J. Math. **5** (1955), 285-309

Application of Tarski's theorem

Tarksi result has many applications in computer science, economics and others. For instance, discrete fixed point theorems have been used to prove the existence of a Nash equilibrium in game theory (The Nash equilibrium of the game is defined as a fixed point of a certain function).

Metric fixed point theory

The concept of metric spaces was introduced in 1906, by the French mathematician Fréchet. Metric fixed point theory came from the Polish mathematician Stefan Banach (1892-1945). Banach fixed point theorem introduced in 1922 is a useful tool in the study of metric spaces. It ensures the presence and uniqueness of fixed points of particular self-maps of metric spaces and gives a constructive approach to find such fixed points.

M. Fréchet, Sur quelques points du calcul fonctionnel. Rend. Circ. Mat. Palermo**22** (1906), 1-74.

Contraction map

Let (X, d) be a metric space, then $T : X \to X$ is said to be a contraction on X if there exists a constant $0 \leq k < 1$ such that

$$d(Tx, Ty) \leq kd(x, y), \text{ for all } x, y \in X.$$

Banach Contraction principle (BCP), 1922

Let (X, d) be a complete metric space and let T be a contraction on X. Then T has a unique fixed point, say $u \in X$. Moreover, for $x_0 \in X$, define $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ (the Picard iteration), then the sequence x_n converges to u, the unique fixed point of T . Also, the rate of convergence is controlled as follows:

$$d(x_n, u) \leq \frac{k^n}{1-k} d(x_0, Tx_0).$$

S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundam. Math. **3** (1922), 133-181.

Suppose (M, d) is a complete metric space and suppose $T : M \to M$ is a mapping for which $T^N = T \circ T \ldots (N$ times) is a contraction mapping for some positive integer $N \geq 2$. Then T has a unique fixed point.
Example. Let $T : [0, 2] \to [0, 2]$ be defined by

$$T(x) = \begin{cases} 0, & \text{if } x \in [0, 1], \\ 1, & \text{if } x \in (1, 2]. \end{cases}$$

Then, $T^2(x) = 0$ for all $x \in [0, 2]$, and so, T^2 is a contraction on $[0, 2]$. Note that T is not continuous and thus not a contraction map.

Applications of the BCP

1. Initial and boundary value problems for ODEs
2. Partial differential equations
3. Newton's Method for finding roots of equations
4. Integral equations
5. Optimization and Game Theory
6. Probability and Stochastic Processes
7. Computer Science and Machine Learning

Initial value problem

$$y' = f(t, y), \ \ t \in [0, T] \tag{1}$$

$$y(0) = y_0 \tag{2}$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $y_0 \in \mathbb{R}$.

$$y(t) = y_0 + \int_0^t f(s, y(s))ds \tag{3}$$

(H_1) The function $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H_2) There exists $k > 0$ such that

$$|f(t, y_1) - f(t, y_2)| \leq k|y_1 - y_2|, \ t \in [0, T], \ y_1, y_2 \in \mathbb{R}.$$

Theorem. Assume that $(H_1), (H_2)$ are satisfied, then problem (1), (2) has a unique solution provided that

$$kT < 1. \tag{4}$$

$$N : C([0, T], \mathbb{R}) \to C([0, T], \mathbb{R})$$

$$Ny(t) = y_0 + \int_0^t f(s, y(s))ds.$$

$$\|Ny_1 - Ny_2\|_\infty \leq kT\|y_1 - y_2\|_\infty.$$

Theorem. Assume that $(H_1),(H_2)$ are satisfied, then problem (1), (2) has a unique solution.

$$\|N^n y_1 - N^n y_2\|_\infty \leq \frac{(kT)^n}{n!}\|y_1 - y_2\|_\infty.$$

$N^n = N \circ \ldots \circ N$ the iteration of order n of N. We have

$$\lim_{n\to+\infty} \frac{(kT)^n}{n!} = 0.$$

Theorem. Assume that $(H_1), (H_2)$ are satisfied, then problem (1), (2) has a unique solution.

$$\|Ny_1 - Ny_2\|_B \leq \frac{1}{\tau}\|y_1 - y_2\|_B,$$

where

$$\|y\|_B = \sup\{e^{-k\tau t}|y(t)| : t \in [0, T]\}$$

the Bielecki norm, and $\tau > 1$.

A. Bielecki, Une remarque sur la méthode de Banach-Cacciopoli-Tikhonov dans la théorie des équations différentielles ordinaires, *Bulletin de l'Académie Polonaise des Sciences. Sér. Mathématique* **4** (1956), 261-264.

Boundary value problem

$$y'' = f(t, y), \ t \in [0, T] \tag{5}$$

$$y(0) = y_0, \ y(T) = y_T \tag{6}$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $y_0, \ y_T \in \mathbb{R}$.

$$y(t) = \left(1 - \frac{t}{T}\right) y_0 + \frac{t}{T} y_T + \int_0^T G(t,s) f(s, y(s)) ds \quad (7)$$

$$G(t,s) = \begin{cases} \frac{s}{T}(t-T) & \text{for } 0 \le s \le t \le T, \\ \frac{t}{T}(s-T) & \text{for } 0 \le t \le s \le T. \end{cases} \quad (8)$$

The function G is called the Green's function.

1. R.P. Agarwal and D. O'Regan, An Introduction to Ordinary Differential Equations, Springer, New York, 2008.
2. G.F. Roach, Green's Functions, Cambridge University Press, Cambridge-New York, 1982.
3. A. Zettl, Recent developments in Sturm-Liouville theory. De Gruyter Studies in Mathematics, 76. De Gruyter, Berlin, 2021.

Properties

1. The function G is continuous on $[0, T] \times [0, T]$
2. $G(0, s) = G(T, s) = 0$
3. The function G is derivable on $[0, T] \times [0, T] \backslash \{(t, s) : s = t\}$.
4. The function G exists provided that the following homogeneous problem has the trivial solution

$$y''(t) = 0, \ \ t \in [0, T] \tag{9}$$

$$y(0) = 0, \ \ y(T) = 0. \tag{10}$$

(H_1) The function $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H_2) There exists $l > 0$ such that

$$|f(t, y_1) - f(t, y_2)| \leq l|y_1 - y_2|, \; t \in [0, T], \; y_1, y_2 \in \mathbb{R}.$$

Set

$$G^* = \sup\{|G(t, s)| : (t, s) \in [0, T] \times [0, T]\}.$$

Theorem. Assume that $(H_1), (H_2)$ are satisfied, then problem (5), (6) has a unique solution provided that

$$lG^* T < 1. \tag{11}$$

$$N : C([0, T], \mathbb{R}) \to C([0, T], \mathbb{R})$$

$$Ny(t) = \int_0^T G(t, s) f(s, y(s)) ds.$$

$$\|Ny_1 - Ny_2\|_\infty \leq lG^* T \|y_1 - y_2\|_\infty.$$

Roots of equations

Let the equation

$$f(x) = 0. \tag{12}$$

Set

$$g(x) = x - \lambda f(x),$$

where λ is a suitable constant such that g is a contraction. If there exists a constance $k \in [0, 1)$ such that

$$|g'(x)| \leq k,$$

then g is a contraction.

Example

Solve the equation

$$x^3 + x - 1 = 0.$$

Define the function

$$g(x) = x - \lambda(x^3 + x - 1).$$

Set $\lambda = 0.5$, then

$$g'(x) = 1 - 0.5(3x^2 + 1).$$

On the interval $x \in [0, 1]$, we have

$$|g'(x)| \leq 0.5 < 1.$$

Hence g is a contraction.

1. **Question A:** Could we find contractive conditions giving existence of a fixed point in a complete metric space without the map is continuous ?
2. **Question B:** Could we have existence of fixed points without the space X is complete ?

Response: Yes for the two questions

Response to Question A:

In 1968, Kannan gave an affirmative answer. He considered this new contraction condition:

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)], \; k \in \left[0, \frac{1}{2}\right).$$

R. Kannan, Some results on fixed points. Bull. Cal. Math. Soc. **62** (1968), 71-76.

Example

Take $X = \mathbb{R}$. Let $T : X \to X$ be defined by

$$T(x) = \begin{cases} 0, & \text{if } x \in (-\infty, 2], \\ \frac{1}{2}, & \text{if } x > 2. \end{cases}$$

Here, 0 is the unique fixed point (even T is not continuous, and so T is not a Banach contraction). Namely, the mapping T is a Kannan contraction with $k = \frac{1}{5}$.

Remark

1. Every contraction map with contraction constant $k < \frac{1}{3}$ is a Kannan map.
2. If $k \in \left[\frac{1}{3}, 1\right)$, then the map is still a contraction (Banach's theorem applies), but it may not satisfy the Kannan condition with $k < \frac{1}{2}$.

Kannan Theorem, 1968

Theorem. Each Kannan contraction mapping on a complete metric space has a unique fixed point.

R. Kannan, Some results on fixed points. Bull. Cal. Math. Soc. **62** (1968), 71-76.

Another version of Kannan Theorem

Theorem. Let (X, d) be a complete metric spaces and $T : X \to X$ be a Kannan contraction mapping, i.e.,

$$d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty), \text{ for all } x, y \in X,$$

where a, b are non-negative real numbers such that $a, b \in [0, 1)$, and $a + b = 1$. Then T has a unique fixed point.

Response to Question B:

There are incomplete metric spaces on which the Banach contraction mapping has a fixed point.

Example Let $X = [0, 1)$ be endowed with the usual distance. Define $T : X \to X$ as $T(x) = \frac{x}{2}$. Here, T is a Banach contraction mapping and has a unique fixed point, even the metric space X is not complete.

Further Questions:

1. How to generalize the Banach contraction ?
2. How to extend the BCP to a large class of various spaces ?
3. Does the converse of BCP hold ?

Meir-Keeler Theorem, 1969

Theorem. Let (X, d) be a complete metric space, and $T : X \to X$. If for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in X, \epsilon \leq d(x, y) < \epsilon + \delta \text{ implies } d(Tx, Ty) < \epsilon,$$

then T has a unique fixed point in X.

A. Meir, E. Keeler, A theorem on contraction mappings, J. Math. Anal. Appl. **28** (1969), 326-329.

Remark

1. Each Meir-Keeler mapping is continuous.
2. Each Banach contraction mapping on a metric space is a Meir-Keeler mapping with
$$\delta = \epsilon\left(\frac{1}{k} - 1\right) \text{ with } k \in (0,1).$$

Example

Let $n \in \mathbb{N}^*$. Endow $X = [0,1] \cup \{3n, 3n+1\}$ with the Euclidean distance. Consider $T : X \to X$ as follows:

$$T(x) = \begin{cases} \frac{x}{2}, & \text{if } x \in [0,1], \\ 0, & \text{if } x = 3n, \\ 1 + \frac{1}{n+2}, & \text{if } x = 3n+1. \end{cases}$$

Meir-Keeler contraction holds (T has a unique fixed point, $u = 0$). But, the Banach contraction is not applicable. Indeed, we have

$$\frac{d(T(3n), T(3n+1))}{d(3n, 3n+1)} = 1 + \frac{1}{n+2} \geq 1.$$

Caristi Theorem, 1976

Theorem. Let (X, d) be a complete metric space and $T : X \to X$. Then T has a fixed point in X provided that there exists a lower semi-continuous function $\phi : X \to [0, \infty)$ such that

$$d(x, Tx) \leq \phi(x) - \phi(Tx), \; x \in X.$$

J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, Transactions of the American Math. Soc., **215** (1976), 241-251.

Remark

1. Recall that the function $\phi : X \to \mathbb{R}$ is said to be lower semi-continuous (l.s.c.) at x, if for any sequence $\{x_n\} \subset X$, with $\lim_{n\to\infty} x_n = x$ and $\liminf_{n\to\infty} \phi(x_n) \geq \phi(x)$, or equivalently if the set $\{x \in X : \phi(x) \leq a\}$ is closed in X for all $a \in \mathbb{R}$.
2. Let $T : \mathbb{R} \to \mathbb{R}$ be defined as

$$T(x) = \begin{cases} 1, & \text{if } x \in (-\infty, 0), \\ 0, & \text{if } x \in [0, \infty). \end{cases}$$

   Then, T is lower semicontinuous at $x = 0$.
3. T is lower semicontinuous at $x = a$ if and only if

$$\liminf_{x\to a} T(x) \geq T(a).$$

Remark

The fixed point is not necessarily unique. Each Banach contraction mapping T on a metric space (X, d) is a Caristi mapping with a function

$$\phi(x) = \frac{1}{1-k} d(x, Tx), x \in X, \text{with } k \in (0, 1).$$

Example

Let $X = [0,1]$ and d be the usual metric. Choose $T : X \to X$ as $Tx = \sqrt{x}$. Take $\phi(x) = 1 - x$. We have

$$d(x, Tx) = \phi(x) - \phi(Tx), \text{ for all } x \in X.$$

Using the Caristi fixed point result, T has two fixed points, $u = 0, 1$. Here, we could not apply the BCP because

$$d(T0, T1) = d(0,1).$$

Remark

1. The function ϕ acts as a kind of potential guiding the motion.
2. The condition says that the distance between a point and its image under T is controlled by the drop in the potential ϕ.
3. Because ϕ cannot decrease indefinitely (bounded below) and is lower semicontinuous, the process stabilizes at a fixed point.

Connections

1. Caristi's theorem is equivalent to Ekeland's variational principle, which is a foundational tool in optimization and variational analysis.
2. It generalizes the Banach contraction principle: if T is a contraction, you can choose an appropriate ϕ to make the Caristi condition holds
3. Caristi's theorem is equivalent to Takahashi's nonconvex minimization theorem. Takahashi's nonconvex minimization theorem is a result in nonlinear analysis that extends fixed-point and minimization principles (in the spirit of Banach's contraction principle) to nonconvex settings. It is particularly useful for proving existence and uniqueness of minimizers without assuming convexity of the domain.

Classical minimization results often require:

1. Convexity of the domain, or
2. Compactness assumptions.

Takahashi's theorem removes convexity while still guaranteeing the existence (and sometimes uniqueness) of a minimizer by imposing a contractive-type condition.

E. Karapinar. Generalizations of Caristi Kirk's theorem on partial metric spaces. Fixed Point Theory and Appl. **2011** (2011), 4, 7 pp.

Ekeland Variational Principle, 1974

Theorem. Let (M, d) be a complete metric space and $\phi : M \to \mathbb{R}^+$ l.s.c. Define:

$$x \prec y \Leftrightarrow d(x, y) = \phi(x) - \phi(y), \text{ for all } x, y \in M.$$

Then, $(M, \prec)$ has a maximal element.

I. Ekeland, On the variational principle. J. Math. Anal. Appl. **47** (1974), 324-353.

Remark

Caristi and Ikeland theorems have a different setting, in fact they are both equivalent. The proof of Caristi and Ekeland theorems is based on discrete technique: Zorn's lemma and Axiom of Choice.

W. Oettli, M. Théra, Equivalents of Ekeland's principle. Bull. Aust. Math. Soc. **48** (1993), 385-392.

Zorn's lemma. Let $(P, \leq)$ be a partially ordered set. If every chain (i.e. every totally ordered subset of P) has an upper bound in P, then P contains at least one maximal element.
Axiom of Choice. Let $\{A_i\}_{i\in I}$ be a family of nonempty sets. There exists a choice function $f : I \to \cup_{i\in I} A_i$ such that

$$f(i) \in A_i \text{ for every } i \in I.$$

The Axiom of Choice (AC) is one of the foundational principles of modern mathematics. It says you can choose an element from each set in a collection even if there is no explicit rule for making the choices.

1. T. Jech, Set Theory, Springer, 3rd millennium edition, Berlin, 2003.
2. Paul R. Halmos, Naive Set Theory, Springer, New York-Heidelberg, 1974

Boyd-Wong Theorem, 1969

Theorem. Let (X, d) be a complete metric space, and suppose that $T : X \to X$ satisfies

$$d(Tx, Ty) \leq \phi(d(x, y)) \text{ for all } x, y \in X,$$

where $\phi : [0, \infty) \to [0, \infty)$ is upper semi-continuous from the right, i.e., for any sequence $t_n \downarrow t \geq 0$

$$\limsup_{n\to\infty} \phi(t_n) \leq \phi(t),$$

and satisfies $\phi(t) < t$ for all $t > 0$. Then T has a unique fixed point.

D.W. Boyd, J.S. Wong, On nonlinear contractions, Proc. American Math. Soc. **20** (2) (1969), 458-464.

Example

1. Let $T : \mathbb{R} \to \mathbb{R}$ be defined as

$$T(x) = \begin{cases} 0, & \text{if } x \in (-\infty, 0), \\ 1, & \text{if } x \in [0, \infty). \end{cases}$$

   Then, T is upper semicontinuous at $x = 0$.
2. T is upper semicontinuous at $x = a$ if and only if

$$\limsup_{x \to a} T(x) \leq T(a).$$

Remark

Matkowski considered the same contraction of Boyd-Wong with a nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ such that

$$\lim_{n\to\infty} \phi^n(t) = 0 \text{ for all } t > 0,$$

where, $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$.

J. Matkowski, Fixed point theorems for mappings with a contractive iterate at a point, Proc. Amer. Math. Soc. **62** (1977), 344-348

Example

Let $X = [0,1] \cup \{2,3,\ldots\}$. We endow on X the complete metric

$$d(x,y) = \begin{cases} |x-y|, & \text{if } x,y \in [0,1], \text{ or } x,y \in \{2,3,\ldots\} \\ 1, & \text{if one of } \ x,y \in [0,1] \text{ and the other in} \{2,3,\ldots\}. \end{cases}$$

Consider $T : X \to X$ as

$$T(x) = \begin{cases} x - \frac{1}{2}x^2, & \text{if } x \in [0,1], \\ x-1, & \text{if } x = 2,3,\ldots. \end{cases}$$

Take

$$\phi(t) = \begin{cases} t - \frac{1}{2}t^2, & \text{if } t \in [0,1], \\ t - 1, & \text{if } t > 1. \end{cases}$$

Here, the Boyd-Wong contraction holds (T has a unique fixed point, $u = 0$). But, as $n \to \infty$,

$$\frac{d(Tn, T0)}{d(n,0)} \to 1.$$

That is, the Banach contraction is not applicable.

Chatterjea's Theorem, 1972

Theorem. Let (X, d) be a complete metric space, and let $T : X \to X$ be a map satisfying,

$$d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)], \text{ for all } x, y \in X,$$

with $0 < k < \frac{1}{2}$. Then T has a unique fixed point.

S.K. Chatterjea, Fixed-point theorems. Comptes Rendus de l'Académie Bulgare des Sciences, **25** (6) (1972), 727-730.

Ciric Theorem, 1974

Theorem. Let (X, d) be a complete metric space, and let $T : X \to X$ be a quasi contraction, that is,

$$d(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

with $0 < k < 1$. Then T has a unique fixed point.

L.B. Ciric, A generalization of Banach's contraction principle. Proc. Am. Math. Soc. **45** (1974), 267-273.

Ordered Metric Space Extensions, Ran and Reurings, 2004

Theorem. Let $(X, \leq)$ be a partially ordered complete metric space, and $T : X \to X$ be monotone, i.e. If $x \leq y$, then $T(x) \leq T(y)$, and there exists $k < 1$ such that:

$$d(Tx, Ty) \leq kd(x, y), \text{ for comparable } x, y.$$

Then T has a unique fixed point.

A.C.M Ran, and .C.B Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations. Proceedings of the American Mathematical Society, **132** (5) (2004), 1435-1443.

A set-valued mapping $T : X \to P(X)$ is said to be contraction if there exists $k \in [0,1)$ such that

$$H(T(x), T(y)) \leq kd(x,y), \text{ for all } x, y \in X.$$

where H is the Hausdorff metric defined by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}.$$

Let $X = \mathbb{R}$, $A = [0,1]$, $B = [2,4]$. Then,

$$\sup_{x\in A} d(x,B) = 2, \ \sup_{y\in B} d(y,A) = 3, \text{ and } H(A,B) = 3.$$

Nadler's Fixed Point Theorem, 1969

Theorem. Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a contraction set-valued mapping. Then, T has a fixed point, that is, there exists $x \in X$ such that $x \in T(x)$.

S.B Jr. Nadler, Multivalued contraction mappings. Pac. J. Math. **30** (1969), 475-488.

Functional Generalizations, Geraghty's Theorem, 1973

Theorem. Let (X, d) be a complete metric space, and let $T : X \to X$ a map satisfying,

$$d(Tx, Ty)) \leq \phi(d(x, y)), \text{ for all } x, y \in X,$$

where $\phi : [0, \infty) \to [0, \infty)$ is a function with :

- $\alpha(t) < 1$ for all $t > 0$.
- ϕ is nondecreasing
- $\phi(t) \to 0$ implies $t \to 0$.

Then T has a unique fixed point x^* and and for any $x_0 \in X$, the sequence $x_n = T^n(x_0)$ converges to x^*.

M. Geraghty, On contractive mappings. Proceedings of the American Mathematical Society, **40** (2) (1973), 604-608.

Random Fixed Point Theorem

Let (Ω, F, P) be a complete probability space, (X, d) a separable complete metric space, $T : \Omega \times X \to X$ a random operator, meaning that for fixed $x \in X$, the map $\omega \to T(\omega, x)$ is measurable, $T(\omega, x)$ is a contraction for each $\omega \in \Omega$, i.e., there exists $0 < k(\omega) < 1$ such that:

$$d(T(\omega, x), T(\omega, y)) \leq k(\omega) d(x, y), \; x, y \in X.$$

Random Banach Contraction

Theorem. If $T : \Omega \times X \to X$ is a random contraction, then there exists a unique random fixed point $x(\omega) \in X$, i.e., a measurable function $x : \Omega \to X$ such that $T(\omega, x) = x(\omega)$, for almost all $\omega \in \Omega$.

R.N. Bhattacharya, and N. Keiding, A generalization of Banach's fixed point theorem for random operators. Proceedings of the American Mathematical Society, **12** (1961), 406-410.

Fuzzy Metric Space: Basic Concepts (Kramosil and Michlek, 1975)

A fuzzy metric space is a generalization of a metric space where distance is described not by a nonnegative real number, but by a fuzzy set indicating how close two points are, based on a degree of truth in $[0,1]$. The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in $[0,1]$, with $u(x) = 0$ corresponding to non-membership, $0 < u(x) < 1$, to partial membership and $u(x) = 1$ to full membership.

Examples of Fuzzy sets

Example 1. The membership function of the fuzzy set of real numbers "close to one" can be defined as

$$A(x) = exp(-\beta(x-1)^2),$$

where β is a positive real number.
Example 2. Let the membership function for the fuzzy set of real numbers "close to zero" defined as follows

$$B(x) = \frac{1}{1+x^3}.$$

Using this function, we can determine the membership grade of each real number in this fuzzy set, which signifies the degree to which that number is close to zero. For instance, the number 3 is assigned a grade of 0.035, the number 1 a grade of 0.5 and the number 0 a grade of 1.

A fuzzy metric space is a triple $(X, M, \star)$, where X is a nonempty set, $M : X \times X \times (0, \infty) \to [0, 1]$ is a function, $\star$ is a continuous t-norm (usually minimum or product), satisfying the following conditions for all $x, y, z \in X, t, s > 0$:

1. $M(x, y, t) = 1$ iff $x = y$,
2. $M(x, y, t) = M(y, x, t)$
3. $M(x, y, t) > 0$,
4. $M(x, y, t) \star M(y, z, s) \leq M(x, z, t + s)$,
5. $M(x, y, \cdot)$ is non-decreasing in t, and

$$\lim_{t \to 0} M(x, y, t) = 0, \quad \lim_{t \to \infty} M(x, y, t) = 1.$$

I. Kramosil, J. Michlek, Fuzzy metric and statistical metric spaces. Kybernetika, **11** (5) (1975), 336-344.

Banach Contraction in Fuzzy Metric Spaces, 1994

Theorem. Let $(X, M, \star)$ be a complete fuzzy metric space, and let $T : X \to X$ be a contraction mapping, i.e., there exists $c \in (0,1)$ such that:

$$M(Tx, Ty, t) \geq M(x, y, ct), \;\; x, y \in X, \;\; t > 0.$$

Then T has a unique fixed point $x \in X$, and the sequence $x_{n+1} = T(x_n)$ converges to x in the sense of the fuzzy metric, i.e., $\lim_{n\to\infty} M(x_n, x, t) = 1, \;\; t > 0.$

A. George, P. Veeramani, On some results in fuzzy metric spaces. Fuzzy Sets and Systems, 64(3) (1994), 395-399.

Applications of fuzzy calculus

Fuzzy calculus (calculus developed for fuzzy numbers and fuzzy-valued functions) is used when uncertainty, vagueness, or imprecision cannot be modeled well by classical calculus. Below are some important applications, grouped by field.

Engineering and Control Systems

1. Fuzzy control systems (e.g., washing machines, air conditioners)
2. Modeling systems with uncertain parameters
3. Stability analysis of systems with fuzzy differential equations
4. Robotics path planning under uncertain environments

Example: Fuzzy differential equations model temperature changes when sensor readings are imprecise.

Economics and Finance

1. Pricing models with uncertain market conditions
2. Investment decision-making with vague risk assessments
3. Modeling inflation, demand, and supply using fuzzy functions
4. Forecasting with incomplete or subjective data

Example: Fuzzy calculus helps model profit functions when costs and revenues are not precisely known.

Medical and Biological Sciences

1. Modeling biological systems with imprecise parameters
2. Drug dosage control under uncertain patient responses
3. Spread of diseases using fuzzy differential equations
4. Medical image processing and diagnosis

Example: Tumor growth models use fuzzy calculus to handle variability in biological responses.

Artificial Intelligence and Machine Learning

1. Fuzzy neural networks
2. Learning systems with uncertain or noisy data
3. Hybrid fuzzy-AI models
4. Pattern recognition and decision-making systems

Example: Fuzzy calculus supports learning algorithms where input features are vague.

Physics and Environmental Science

1. Heat transfer and fluid flow with uncertain boundary conditions
2. Climate modeling with imprecise environmental data
3. Quantum systems with fuzzy states
4. Pollution dispersion modeling

Example: Fuzzy partial differential equations model air pollution when emission data is uncertain.

Social Sciences

1. Population dynamics with uncertain growth rates
2. Decision-making models involving human judgment
3. Risk assessment and policy analysis

A probabilistic metric space is a triple (X, F, τ), where X is a non-empty set, $F : X \times X \to \Delta$, where Δ is the set of distribution functions (non-decreasing, left-continuous functions $F : \mathbb{R} \to [0, 1]$ with $F(0) = 0$, and $\lim_{t\to\infty} F(t) = 1$), τ is a triangle function (generalized form of addition). $F(x, y)(t)$ gives the probability that the distance between x and y is less than or equal to a value t.

Probabilistic Banach Contraction Theorem

Theorem. Let (X, F, τ) be a complete probabilistic metric space. Let $T : X \to X$ be a mapping such that there exists a distribution function φ satisfying:

1. $F(Tx, Ty)(t) \geq \varphi(t), \; x, y \in X, t > 0$,
2. $\varphi(t) > 0$ for $t > 0$,
3. $\varphi(\cdot)$ is strictly increasing, and
4. $\lim_{t \to \infty} \varphi(t) = 1$.

Then T has a unique fixed point $x \in X$, and the iteration $x_{n+1} = T(x_n)$ converges to x in the probabilistic sense.

B. Schweizer, and A. Sklar, Statistical metric spaces. Pacific Journal of Mathematics, **10** (1960), 314-334.

Banach Contraction Theorem in b-metric space

A b-metric space (or a quasi-metric space) is a pair (X, d), where $d : X \times X \to [0, \infty)$ satisfies for all $x, y, z \in X$:

1. $d(x, y) = 0 \Leftrightarrow x = y$ (identity of indiscernible),
2. $d(x, y) = d(y, x)$ (symmetry),
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$, for some $s \geq 1$.

Example

Let $X = C([0,1],\mathbb{R})$ be the space of continuous functions, and $d : X \times X \to \mathbb{R}_+$ the map defined by

$$d(y,z) = \sup_{t\in[0,1]} |y(t) - z(t)|^2.$$

Then d is a b-metric with $s = 2$ and (X,d) is a b-metric space.

Example

Let $X = \mathbb{R}$, and $d : X \times X \to \mathbb{R}_+$ the map defined by

$$d(x_1, x_2) = (x_1 - x_2)^2.$$

Then d is a b-metric with $s = 2$ and (X, d) is a b-metric space.

Example

Let $0 < p < 1$,

$$X = L^p([0,1],\mathbb{R}) = \left\{x : [0,1] \to \mathbb{R} : \int_0^1 |x(t)|^p dt < \infty\right\},$$

and $d : X \times X \to \mathbb{R}_+$ the map defined by

$$d(x_1, x_2) = \left(\int_0^1 |x_1(t) - x_2(t)|^p dt\right)^{\frac{1}{p}}.$$

Then d is a b-metric with $s = 2^{\frac{1}{p}} - 1$ and (X, d) is a b-metric space.

Banach Contraction Principle in b-metric Spaces, 1993

Theorem. Let (X, d) be a complete b-metric space with constant $s \geq 1$, and let $T : X \to X$ be a mapping such that:

$$d(Tx, Ty) \leq kd(x, y), \; x, y \in X,$$

for some constant $k \in [0, 1)$. Then:

1. T has a unique fixed point $x \in X$,
2. The sequence $x_{n+1} = T(x_n)$ converges to x in the b-metric.

S. Czerwik, Contraction mappings in b-metric spaces, Acta Mathematica et Informatica Universitatis Ostraviensis, **1** (1) (1993), 5-11.

Remark: A b-metric is not necessary continuous

Let $X = \mathbb{R}$ and define the map $d : X \times X \to \mathbb{R}_+$ by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ |x| + |y| + 1, & \text{if } x \neq y. \end{cases}$$

One can check that d is a b-metric with $s = 2$. However, this map is not continuous. Indeed, let

$$x_n = \frac{1}{n}, \; y_n = 0, \; n \in \mathbb{N}^*.$$

Then

$$(x_n, y_n) \to (0,0), \; d(x_n, y_n) = 1 + \frac{1}{n} \to 1,$$

but $d(0,0) = 0$. Hence d is not continuous.

Banach Contraction Principle in Generalized Banach Spaces

Definition

Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^n$ with the following properties:

(i) $d(y_1, y_2) \geq 0$ for all y_1, $y_2 \in X$, and if $d(y_1, y_2) = 0$, then $y_1 = y_2$;

(ii) $d(y_1, y_2) = d(y_2, y_1)$ for all y_1, $y_2 \in X$;

(iii) $d(y_1, y_2) \leq d(y_1, w) + d(w, y_2)$ for all y_1, y_2, $w \in X$.

A set X with a vector-valued metric d is called a *generalized metric space*. In this space, the notions of Cauchy sequence, convergence, completeness, and open and closed sets are similar to those in usual metric spaces. Here, if $y_1, y_2 \in \mathbb{R}^n$, where $y_1 = (y_{1,1}, y_{1,2}, \cdots, y_{1,n})$ and $y_2 = (y_{2,1}, y_{2,2}, \cdots, y_{2,n})$, by $y_1 \leq y_2$ we mean $y_{1,i} \leq y_{2,i}$ for $i = 1, 2, \cdots, n$. The pair (X, d) is a generalized metric space with

$$d(y_1, y_2) := \begin{pmatrix} d_1(y_1, y_2) \\ \vdots \\ d_n(y_1, y_2) \end{pmatrix}.$$

Notice that d is a generalized metric on X if and only if $d_i, \; i = 1, 2, \cdots, n$, are metrics on X.

Similarly, a *vector valued norm* on a linear space X is a mapping $\|\cdot\| : X \to \mathbb{R}^n_+$ with

(*i*) $\|y\| = 0$ only for $y = 0$,

(*ii*) $\|\lambda y\| = |\lambda| \|y\|$ for $y \in X$ and $\lambda \in \mathbb{R}$,

(*iii*) $\|y_1 + y_2\| \leq \|y_1\| + \|y_2\|$ for every $y_1, y_2 \in X$.

Associated to a vector valued norm $\|\cdot\|$ is a *vector valued metric* $d(y_1, y_2) := \|y_1 - y_2\|$, and we say that $(X, \|\cdot\|)$ is a *generalized Banach space* if X is complete with respect to d.

Definition

A square matrix with real entries is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, all the eigenvalues of M are in the open unit disc $|\lambda| < 1$ for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$, where I denotes the identity matrix in $\mathcal{M}_{n\times n}(\mathbb{R})$.

Theorem

Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$; *the following assertions are equivalent:*

(a) M *is convergent to zero;*

(b) $M^k \to 0$ *as* $k \to \infty$;

(c) *The matrix* $(I - M)$ *is nonsingular and*

$$(I - M)^{-1} = I + M + M^2 + \cdots + M^k + \cdots ;$$

(d) *The matrix* $(I - M)$ *is nonsingular and* $(I - M)^{-1}$ *has nonnegative elements.*

Example

Some examples of matrices that are convergent to zero:

(i) $A = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ *and* $a + b < 1$;

(ii) $A = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ *and* $a + b < 1$;

(iii) $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ *and* $|a - b| < 1,\ a > 1,\ b > 0$.

Definition

Let (X, d) be a generalized metric space. An operator $T : X \to X$ is called contractive associated with the above d on X, if there exists a convergent to zero matrix M such that

$$d(T(x), T(y)) \leq Md(x, y) \text{ for all } x, y \in X.$$

Perov's fixed point theorem, 1964

Theorem

Theorem. *Suppose that* (X, d) *is a complete generalized metric space and* $T : X \to X$ *is a contractive operator with Lipschitz matrix* M. *Then* T *has a unique fixed point* u, *and for each* $u_0 \in X$,

$$d(T^k(u_0), u) \leq M^k(I - M)^{-1} d(u_0, T(u_0)) \text{ where } k \in \mathbb{N}.$$

1. A. I. Perov, On the Cauchy problem for a system of ordinary differential equations, *Pviblizhen. Met. Reshen. Differ. Uvavn.* **2** (1964), 115-134. (in Russian).
2. J. R. Graef, J. Henderson, A. Ouahab, *Topological Methods for Differential Equations and Inclusion*, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, 2019.

Characterization of metric completeness (Inverse of the BCP)

The famous BCP holds in complete metric spaces, but as said before in the response of question B, completeness is not a necessary condition. Indeed, we said there are incomplete metric spaces on which every contraction mapping has a fixed point. On the other hand, contractions on incomplete metric spaces may fail to have fixed points.

Remark

This principle does not characterize the metric completeness of a metric space. Characterizations of metric completeness have been discussed in several works, that is, to present various circumstances in which fixed point results imply completeness.

Characterization 1

Theorem: A metric space (X, d) is complete if and only if every Caristi mapping on (X, d) has a fixed point.

W.A. Kirk, Caristi's fixed point theorem and metric convexity, Colloquium Mathematicum, **36** (1) (1976), 81-86.

Characterization 2

Theorem: A metric space (X, d) is complete if and only if every Kannan mapping on (X, d) has a fixed point.

P.V. Subrahmanyam, Completeness and fixed-points, Monatshefte fur Mathematik, **80** (4) (1976), 325-330.

Conclusion

- Many extensions of BCP
- Applicable to nonlinear analysis, integral equations, and ODEs
- Characterizations of metric completeness exist via fixed point theorems

Recognition to Banach

On April 3rd, 2012 r. National Bank of Poland issued three coins (2 PLN, 10 PLN and 200 PLN) commemorating Stefan Banach.

1. The coin 200 PLN was made in gold and was devoted to the Banach Fixed Point Theorem.

1. The coin 10 PLN coin was made in silver and showed the relation between linear mappings on Banach spaces.
2. The coin 2 PLN presented inequality characterizing bounded linear mappings on Banach spaces.

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Thank you for your attention